I.) Plasma on a Back-of-Envelope

→ Basic Ideas

→ What is a plasma?

- dilute gas
- charged particles – i.e. electrons + ions, usually protons – but net neutral
- long range Coulombic interaction
- screening

→ Recall – dilute gas

- basic ordering: (collisional transport)

$$d < \bar{r} < l_{mfp} < L$$

d → range force (i.e. radius of hard sphere)

$\bar{r}$ → mean interparticle spacing

l_{mfp} → mean free path

L → system size

or: long mean free path regime:

$$d < \bar{r} < L < l_{mfp}$$

some key $\left\{\begin{array}{l}\text{orderings} \\ \text{estimates}\end{array}\right.$:

① $d^3 n = d^3/\bar{r}^3 \ll 1$

→ diluteness – particles ≈ free, mostly non-interacting

②
$$l_{mfp} \sim 1/n\sigma \sim 1/n\pi d^2$$
$$\sim \bar{r}\,(\bar{r}/d)^2$$

∴
$$l_{mfp}/\bar{r} \sim (\bar{r}/d)^2$$

diluteness assures l_{mfp} exceeds particle spacing (i.e. gas vs liquid, etc.).

$$l_{mfp}/d \sim (\bar{r}/d)^3$$

diluteness assures l_{mfp} exceeds range of force.

③ $\nu_c \sim V_{th}/l_{mfp} \sim V_{Th}\, n\sigma$

~ defines collision frequency.

For l_{mfp}:

particles

i.e.

particle + interaction cylinder

$$V_{IC} \approx \sigma L$$

so

$\alpha \equiv$ # collisions in cylinder of length L

$$\alpha = n\sigma L$$

so mean length between collisions

$$l_{mfp} \cong L/\alpha \simeq (1/n\sigma)$$

or

$$\left(\frac{d\alpha}{dL}\right)^{-1} \simeq 1/n\sigma \simeq l_{mfp}.$$

$$\boxed{l_{mfp} \sim 1/n\sigma}$$

(4) basic diffusivity:

$$D \sim v_{Th}\, \ell_{mfp} \sim v_{Th}^2 / \nu_c$$

Now, for plasma:

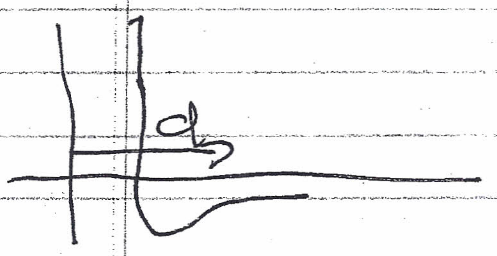
- force is Coulomb

i.e. $V = e\phi \sim 1/r$

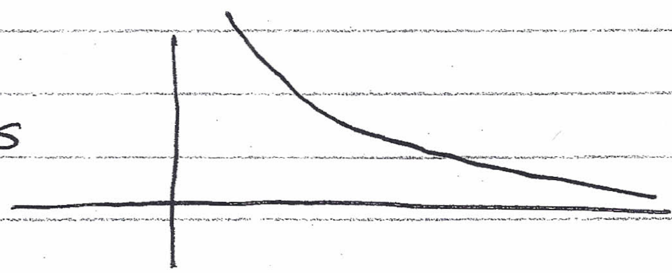
$\Rightarrow$ no scale associated!

$\Rightarrow$ long range!

i.e. contrast hard sphere



vs



$\therefore$ no d.

$\Rightarrow$ screening occurs!

so now, fundamental collisional scale ordering is:

$$\boxed{\bar{r} < \lambda_D \sim \lambda_D < \ell_{mfp} < L}$$

$\bar{r} \sim 1/n^{1/3}$ → mean inter-particle spacing

λ_D ~ Debye length → key scale in plasma

ℓ_{mfp} ~ mean free path

$\ell_{mfp} \sim 1/n\sigma$

Key: cross-section, with long range interaction

L → system size

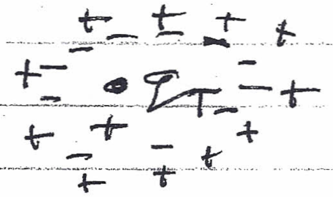
Key Point: Debye Length λ_D ?

For Debye Length:

in plasma

q_T

test charge



plasma adjusts to screen charge

$\Longleftrightarrow$ needs energy

$\rightarrow T$

$1/r \rightarrow e^{-r/\lambda_D}/r$

$$\nabla^2\phi = -4\pi\rho$$

$$= -4\pi n_0|e|\left[\frac{\delta n_i}{n_0} - \frac{\delta n_e}{n_0}\right] + 4\pi q_T\,\delta(x - x_T)$$

media/plasma response

bare charge

$$\delta n_i/n_0 = \exp\left[-|e|\phi/T_i\right]$$

$k_B \rightarrow 1$ eV

$$\delta n_e/n_0 = \exp\left[|e|\phi/T_e\right]$$

so, noting neutrality (net):

$$\nabla^2\phi = -4\pi n_0|e|\left[+\cancel{1} - \frac{|e|\phi}{T_i} - \cancel{1} + \frac{|e|\phi}{T_e}\right]$$

$$\nabla^2\phi = 4\pi n_0 e^2\left(\frac{1}{T_e} + \frac{1}{T_i}\right)\phi$$

$$\phi = \exp\left[-r/\lambda_D\right]/r$$

$$\boxed{1/\lambda_D^2 = 4\pi n_0 e^2\left(1/T_e + 1/T_i\right)}$$

Screening Debye Length

$$\lambda_D^2 = \left[4\pi n_0 e^2\left(1/T_i + 1/T_e\right)\right]^{-1}$$

$\Rightarrow$

① key feature of plasma:

$$\boxed{n\lambda_D^3 \gg 1}$$

$\rightarrow$ large number of particles in Debye sphere

$$\boxed{\lambda_D > \bar{r} \sim 1/n^{1/3}}$$

Why: $\rightarrow$ diluteness!

$\rightarrow$ $T > e^2/r$

i.e. thermal energy must exceed electrostatic energy

$\Rightarrow$ $\left\{ \begin{array}{l} \text{dilute} \\ \text{plasma} \end{array} \right.$, not crystal. !

check: $T > e^2/\bar{r}$

$$\frac{n T \bar{r}}{n e^2} > 1$$

$$\Rightarrow \lambda_D^2 \bar{r} n > 1$$

$$\lambda_D^2 > \bar{r}^2$$ ✓

N.B.: orders: $\bar{r} < \lambda_D$.

Also: plasma classical;

why: Thermal fluctns exceed QM fluctuations, in energy.

Now: thermal $\rightarrow$ T

– QM $\epsilon \sim p^2/2m$

$\sim \hbar^2 k^2/2m$

$\sim \hbar^2/\bar{r}^2 2m \sim \hbar^2 n^{2/3}/m$

$\Rightarrow$ $\boxed{T \gg \hbar^2 n^{2/3}/m}$

and have:

$T \gg e^2 n^{1/3}$ $\rightarrow$ diluteness

so, for dilute plasma:

Coulomb > Quantum.

$e^2 n^{1/3} > \hbar^2 n^{2/3}/m$

$\Rightarrow$ $e^2 n^{1/3}/\hbar^2 n^{2/3}/m \simeq \dfrac{me^2}{\hbar^2 n^{1/3}}$

$\simeq \dfrac{\bar{r}}{a_B} \gg 1$

$\left\{ \begin{array}{l} \text{mean interparticle} \\ \text{spacing must exceed} \\ \text{Bohr radius} \end{array} \right.$

a_B $\downarrow$ Bohr radius

where: $a_B = \dfrac{4\pi\hbar^2}{me^2}$ → Bohr radius $\sim 1\,nm$.

check: $\dfrac{\hbar^2}{2m\,a_B^2} \sim \dfrac{e^2}{a_B}$

$a_B \sim \hbar^2/me^2$

so conditions for plasma:
(classical)

$$\boxed{\begin{array}{l} n\lambda_D^3 \gg 1 \\ \lambda_D^3/a_B^3 \gg 1 \end{array}}$$

and so have scale ordering:

$$\boxed{\bar{r} < \lambda_D < l_{mfp} < L}$$

Frequencies / Resonances

- much of plasma physics deals with waves / instabilities
- collective resonances

$$\underline{\nabla} \cdot \underline{D} = 4\pi \rho_{ext}$$

$$\underline{D} = \underline{E} + 4\pi \underline{P} = \epsilon(\omega) \underline{E}$$

$\underline{P}$ ↓ polarization; $\epsilon(\omega)$ ↑ dielectric fctn.

N.B. in most generality;

$$\epsilon(\omega) \to \epsilon(\omega, k)$$

and, say, electron polarization:

$$\underline{P} = n e \delta \underline{x} \qquad (e<0)$$

i.e. consider high frequency wave oscillation, so electron inertia low.

$$m_e \frac{d^2 x}{dt^2} = e \underline{E}$$

$$\left(\underline{E} = \underline{E}_0 e^{-i\omega t} \right)$$

$\Rightarrow$ $\to$ excursion

$$-\omega^2 m_e \delta \underline{x} = e \underline{E}$$

$\Rightarrow$

$$\delta \underline{x} = -e\, \underline{E}/\omega^2 m_e$$

so

$$4\pi \underline{P} = -\frac{4\pi n_0 e^2}{m_e \omega^2} \underline{E}$$

$$= -\omega_{pe}^2/\omega^2$$

$$\boxed{\omega_{pe}^2 = \frac{4\pi n_0 e^2}{m_e} \to \text{plasma frequency}}$$

$\to$ space charge oscillation wave

$\to$ $\delta n \to \delta \underline{E} \Rightarrow$ restoring force.

so

$$\underline{D} = \underline{E} + 4\pi \underline{P} = \left(1 - \frac{\omega_p^2}{\omega^2}\right) \underline{E}$$

$$\epsilon(\omega) = 1 - \omega_p^2/\omega^2$$

so

$$\underline{\nabla} \cdot \underline{D} = \left(1 - \omega_p^2/\omega^2\right) \underline{\nabla} \cdot \underline{E} = 4\pi \rho_{ext}$$

so → for $\rho_{ext} = \rho_{ext}(\omega \sim \omega_{pe})$

⇒ $\underline{E}$ response in plasma is large ⇒ $\epsilon \to 0$

– $\boxed{\text{collective resonance or mode}}$

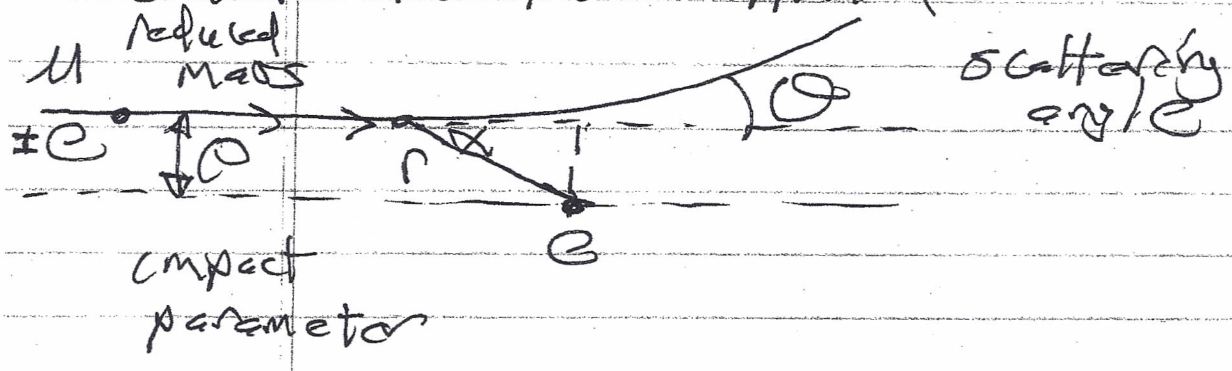
– origin is space charge separation ⇒ restoring force.

identifies: fundamental!

$$\boxed{\begin{array}{l} \omega_p^2 = 4\pi n q^2/M \\ \text{plasma frequency} \end{array}}$$

Transport $\Leftrightarrow$ Coulomb Collisions

→ Consider familiar collision:



– what is cross section?

in particular, seek cross section for weak deflection ⟺ "momentum transfer cross-section"

i.e. more glancing collisions occur...

– of course, central force, so $|p|$ conserved, but direction changed

$$\mu \Delta V_\perp = \Delta p_\perp = \int_{-\infty}^{+\infty} dt \, F_\perp$$

$$= \int_{-\infty}^{+\infty} dt \, \frac{e^2}{r^2} \sin\alpha$$

$$= \int_{-\infty}^{+\infty} dt \, \frac{e^2}{r^2} \frac{b}{r}$$

$$\Delta p_\perp = e^2 \int_{-\infty}^{+\infty} \frac{\rho \, dt}{(\rho^2 + v^2 t^2)^{3/2}} \sim e^2 \rho \int_{-\infty}^{+\infty} \frac{1}{\rho^3} \frac{dt}{\left(1 + \frac{v^2 t^2}{\rho^2}\right)^{3/2}}$$

$$\sim e^2/\rho v$$

but $\Delta p_\perp \sim \mu v \sin \theta$
$\sim \mu v \theta$

so deflection angle:

$$\boxed{\theta \sim e^2/\mu v^2 \rho}$$

then for cross section:

$$d\sigma = \underbrace{\rho \, d\rho}_{\text{area of interaction cylinder}} = d(\rho^2) = d\left(e^2/\mu v^2 \theta\right)^2$$

i.e. note key point: cross section heavily weights weak deflections

$$d\sigma \simeq \left(\frac{e^2}{\mu v^2}\right)^2 \frac{d\theta}{\theta^3}$$

Now: $d\sigma = \left(\frac{e^2}{\mu v^2}\right)^2 \frac{d\theta}{\theta^3}$

weak deflection divergence

n.b.: small $\theta \Leftrightarrow$ large ρ

$\Rightarrow$ long range character of Coulomb force!

$\Rightarrow$ screening, long range cut-off is very relevant.

Now for momentum transfer cross-section need take out collisions with no transfer, i.e. $\rightarrow$ take out forward scattering.

$$d\sigma_t = (1-\cos\theta)\, d\sigma$$

$$\simeq \theta^2 \left(\frac{e^3}{\mu v^2}\right)^2 \frac{1}{\theta^3}$$

so $$d\sigma_t \simeq \left(\frac{e^2}{\mu v^2}\right)^2 \frac{1}{\theta}$$

$\therefore$ $$\boxed{\sigma_T \cong \left(\frac{e^2}{mv^2}\right)^2 \ln\left(1/\theta_0\right)}$$

divergence - low θ

- Coulomb cross-section, Rutherford
- θ_0 is small angle cut-off

now low $\theta \Leftrightarrow$ large ρ

$\Rightarrow$ small angle cut-off set by large ρ

$\therefore$ largest ρ can be λ_D $\leftarrow$ screening limited!

Now,

$$\theta \sim e^2/mv^2\rho$$

so

$$\boxed{\theta_0 \sim e^2/mv^2\lambda_D}$$

screening cut-off

So $\ln \Lambda = \ln(1/\theta_0) = \ln(T\lambda_D/e^2)$

$\hookrightarrow$ ($\ln \Lambda$)

Coulomb Logarithm (can resolved by $G \to LB$)

$$\boxed{\sigma_t \sim \left(\frac{e^2}{T}\right)^2 \ln \Lambda}$$ $\to$ effective cross section

$\Rightarrow$

$$\boxed{\sigma_t \sim \bar{r}^2 \left(e^2/\bar{r}T\right)^2 \ln \Lambda}$$

Coulomb cross section

Note: $\left(e^2/\bar{r}T\right)^2 \to \left(\left[1/n\lambda_D^3\right]^{2/3}\right)^2 \sim \dfrac{\bar{r}^4}{\lambda_D^4}$

$\left(\bar{r}^2/\lambda_D^2\right)^{-2}$

$\sim 1/\left(n\lambda_D^3\right)^{4/3}$

$$\boxed{\sigma_t \sim \bar{r}^2 \left(1/n\lambda_D^3\right)^{4/3} \ln \Lambda}$$

Now,

$$l_{mfp} \sim 1/n\sigma_t$$

$$\sim 1/n\,\bar{r}^2\left(\frac{e^2}{\bar{r}T}\right)^2 \ln\Lambda$$

$$\sim \bar{r}\,(n\lambda_D^3)^{4/3}/\ln\Lambda$$

$$\boxed{l_{mfp} \simeq \bar{r}\,(\lambda_D/\bar{r})^4/\ln\Lambda}$$

so $\quad l_{mfp} \simeq \bar{r}\,(\lambda_D/\bar{r})^4/\ln\Lambda$

$$\frac{l_{mfp}}{\lambda_D} \sim \left(\frac{\lambda_D}{\bar{r}}\right)^3/\ln\Lambda$$

$$\sim n\lambda_D^3/\ln\Lambda$$

as $\quad n\lambda_D^3 \gg \ln\Lambda$

$$\boxed{l_{mfp} \gg \lambda_D.}$$

consistent with screening

so

$$\boxed{\bar{r} \ll \lambda_D < l_{mfp} \ll L}$$

collisional plasma ordering.

Note: $\dfrac{l_{mfp}}{\lambda_D} \cong \dfrac{\bar{r}}{\lambda_D}\left(\dfrac{\lambda_D}{\bar{r}}\right)^4 \Big/ \ln\Lambda$

$$\cong \left(n\lambda_D^3\right) \Big/ \ln\Lambda$$

Now, further points about transport:

- apart from $\ln\Lambda$, no mass, M scaling in V_t, l_{mfp}.

- so $\nu_{col} \sim M^{-1/2}$, via V_{th}.
  $\tau_{col} \sim M^{1/2}$

$\underline{\underline{so}}$ $\dfrac{\tau_{e,e}}{\tau_{ii}} \sim \left(M_e/M_i\right)^{1/2} \ll 1.$

then, as before (gases):

→ <u>thermal conductivity</u>

$$\lambda \sim n \; v_{Th} \; l_{mfp}$$

$n \to C_v$, $v_{Th} \to$ larger for electrons, $l_{mfp} \to$ indep. M

so electrons control thermal conduction.

→ <u>viscosity</u>

$$\eta \sim M_i n \, v_{Th_i} \, l_{mfp}$$

$$l_{mfp} \sim 1/n\sigma$$

ions control flow

contrast: $\eta_e \sim M_e n \, v_{The} \, l_{mfp}$

$$\eta \sim M \, v_{Thi} / \sigma_t$$

$$\boxed{\eta \sim (M_i T)^{1/2} / \sigma_t}$$